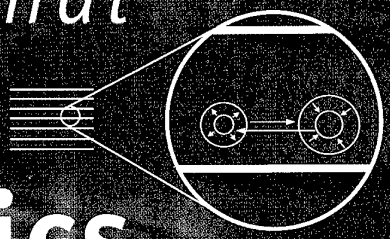
First International Workshop on Thermoacoustics



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LINEAR THERMOACOUSTICS

N. Rott

1. Introduction

The first theory of sound propagation which has fully accounted for the effects of heat conduction on the speed of sound was given by Kirchhoff [1] in 1868. Starting with the energy equation for an ideal gas which was proposed in 1842 by the German physician Robert Mayer, Kirchhoff was first to formulate a version of this equation in which density fluctuations were the only non-thermal effects included and thus the sole "cause" of a correction to the Newtonian value of the sound speed which was initiated by Laplace in 1821. The linearization of this equation permitted its combination with the small-disturbance approximations of the equations of motion, and a theory of sound propagation was established.

In a completely unbounded atmosphere of an ideal gas, Kirchhoff reconfirmed what Stokes has already found, looking at the effect of friction alone: corrections to the Laplacian value of the sound speed are negligibly small for so-called "perfect" (actually, monatomic) gases. Kirchhoff proceeded, however, to consider the problem of sound propagation in ducts. This is the question we wish to re-examine here.

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A critical examination of Kirchhoff's derivation has shown that had he started with a form of the energy equation wherein the pressure fluctuations are the only ones which have non-thermal origin, then his final results would have looked quite differently. This conclusion seems to be incongruous as it implies that beginning with the same set of "exact" equations, different systems of linearized equations can be constructed. The aim of this paper is to show that this conclusion is actually correct, and to prove this point, two possible linear systems will be constructed. One is Kirchhoff's original solution and the other is a competing version.

The examples will show why these results are *not* paradoxical. When facing a complicated system of exact non-linear equations, one can notice that the *order* in which linear approximations are combined influences the outcome. A second point of great importance for these derivations is the effect of *averaging*, which one is inclined to consider as a linear operator. However, conclusions drawn from such operations are "irreversible".

Upon establishment of a linear system, the task which remains is the verification of the physical "mode" that is best described by the results that have emerged. We will find the new method of linearization to be useful for a wider range of phenomena than the old.

Next we will turn our attention to "genuine" thermoacoustic problems where sound propagates in a gas with non-uniform temperature. In the example first treated in 1949 by the noted Dutch physicist Hendrik Kramers (1894-1952), oscillations of a gas column with a prescribed axial temperature distribution were considered [2]. Kramers constructed a very complicated small disturbance theory for the solution but ultimately

expressed doubts in the possible success of linear solutions in the explanation of fundamental phenomena. The particular circumstances which caused special difficulties in his pioneering work will be noted later.

In 1969, I published a paper on this subject [3] which showed considerable advantages in the use of the pressure as the primary state variable in these linearized systems. The developments in the field have proven that as predicted by Kramers, linear theory can not provide complete solutions to problems in thermoacoustics. However, here we wish to reconsider the question: how far can linear theory go? And in particular, we wish to reconsider the question of the "best" small disturbance variables, by returning to the discussion of Kirchhoff's linearization and its alternative(s). It will be shown that a more effective linear system will use not only the pressure but also an "associated" volume flux. We will conclude that the search for the "best" linear approximation is not over.

We postulate that rigorous solutions of linear systems must be homogeneous, i.e., capable of free oscillations, which however may be damped or excited, with the question of the stability limit and the existence of complex-periodic solutions as the central problem. We have to realize that in a basic (but admittedly artificial) equilibrium situation where the sound speed varies along a tube, periodic linear solutions in general *do not exist*. Exceptions can be constructed for limiting conditions, like the one first considered by Kramers, where a gas column is divided into two sections, each with a constant temperature, and with the ratio of the absolute temperatures between the two sections assumed to be very high. Then the time spent by a wave in the hot part is negligibly small compared to the

time it needs to traverse the cold part. In this limit, periodic solutions will be found.

Kirchhoff was first to face the problem of incorporating the energy equation of the ideal gas *and* the attendant molecular-kinetic effect, i.e., heat conduction, in the theory of sound propagation. Before him, the momentum equation and the use of friction alone was thought to give adequate foundations. We will continue a tradition established by Kirchhoff and put the energy equation at the beginning, and will proceed by treating linear friction and heat conduction on a strictly equal footing. This was done before but without arriving to some ultimate conclusions that will be presented here.

We also follow Kirchhoff by giving the search for homogeneous solutions the highest priority, even though many related physical effects of greatest importance remain excluded. First among them is the celebrated theory by Rayleigh [4] of "acoustic streaming" (1883), which is a consequence of the full Navier-Stokes equations together with selective statements based on the energy equation. Unresolved problems in connection with this well-observed effect will be briefly noted.

In retrospect it is understandable why Kirchhoff never doubted that his theory is fully adequate. He gave one way of linearization and it is only natural to assume that all other methods will give results that are only trivially different. Actually only efforts to go beyond Kirchhoff's original theory and to treat specific thermoacoustic problems gave the incentive to reconsider the question whether the energy equation is given its proper treatment in acoustics in particular and in fluid mechanics in general.

2. The energy equation

Beginning his analysis, Kirchhoff states that in a gas without heat conduction (and incidentally, without friction) the relation $c_V \rho dp - c_p p d\rho = 0$ holds. In case there is heat conduction, the expression shown is not equal to zero but gives (in Kirchhoff's own words) "something more complicated". We quote a modern version of Kirchhoff's formulation, and intend to discuss both the basic contents which remained unchanged ever since and some details which reflect more recent changes of usage:

$$\mathrm{div}(\kappa \mathrm{grad} T) = \frac{T}{p}\left(c_V \rho \frac{Dp}{Dt} - c_p p \frac{D\rho}{Dt} \right) \qquad (1)$$

First we note that Kirchhoff does *not* introduce a "mechanical equivalent of heat" in establishing (1), but assumes matter-of-factly that mechanical and thermal energy are measured using the same units. In a recent (1953) presentation of the energy equation of the ideal gas by Howarth [5], the universal acceptance of this practice was hailed as progress achieved by international agreements in the mid-twentieth century. I did not make historical studies that would prove Kirchhoff's priority in this "modern" view, but his paper certainly marks an important progress compared to some confusing presentations of the energy equation in the mid-nineteenth century.

However, the statement of (1) is incomplete. The heat content of a fluid element is also augmented by dissipation, causing what can justifiably be called the "Joule heating". It is proportional to the coefficient of

viscosity μ and contains terms proportional to the *square* of the velocity derivatives. For this reason, the whole term falls victim to the linearization process which will be applied presently. Nevertheless, an expression of the energy equation including Joule heating is given here, for completeness, including some changes of (1) which brings it to a more familiar form:

$$\frac{1}{a^2}\frac{Dp}{Dt} - \frac{D\rho}{Dt} = \frac{1}{c_p T}[\mathrm{div}(\kappa \mathrm{grad} T) + \mu J] \qquad (2)$$

with $c_p/c_v = \gamma$ and $\gamma p/\rho = \gamma RT = a^2$, where a is the speed of sound, and J is the dissipation function, given (with the e_{ii} omitted) by

$$J = e_{12}^2 + e_{23}^2 + e_{31}^2, \qquad e_{ik} = \partial u_i / \partial x_k + \partial u_k / \partial x_i \qquad (3)$$

Returning to (1), we note that in Kirchhoff's paper the partial derivative $\partial/\partial t$ is used instead of the total (substantial) derivative D/Dt. The use of D/Dt makes the term in (1) "exact" and shows that linearization is achieved by the operation $D/Dt \rightarrow \partial/\partial t$.

Furthermore, Kirchhoff writes on the left-hand side of (1) κ-times the Laplacian of the temperature instead of the more elaborate expression $\mathrm{div}(\kappa \mathrm{grad} T)$. For an ideal gas κ is a function of the temperature, and it would be possible to introduce a new temperature function defined by $dT_{new} = \kappa(T)\, dT$ into the analysis. This would lead to serious changes in the theory which are essentially unexplored. However, for sufficiently small variations of the temperature range it will suffice to use $\kappa \nabla^2 T$, and an assumption first introduced by Kramers [2]. For very strong axial variations of the temperature in one direction (x, say), the coefficient κ will be

assumed to be constant in a cross-section but dependent on the (mean) temperature as a function of the coordinate x. An analogous assumption will be made later for the temperature dependence of the viscosity.

Finally, in Kirchhoff's original formula, the first factor on the right-hand side of the equation corresponding to (1) is $1/\alpha p_0$, where α is the "coefficient of compressibility" of the gas. We will restrict our attention to ideal gases with the equation of state $p = RT\rho$, and use only the absolute temperature T. The factor in question then becomes T/p. This expression is used in (1); it is in agreement with results of the article by Howarth [5], to which we refer as the proper authority for modern usage.

For the purposes of the present paper it will be sufficient (and from an advanced point of view, necessary) to assume constant specific heats.

We proceed to quote more formulations of the exact energy equation (2), obtained with the help of the relations:

$$\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{1}{p}\frac{Dp}{Dt} - \frac{1}{T}\frac{DT}{Dt} = -\mathrm{div}\,\underline{u} \tag{4}$$

i.e., the equation of state and of the continuity equation, which are two "trivial" statements in the present context. Eliminating the density one finds

$$\rho c_p \frac{DT}{Dt} - \frac{Dp}{Dt} = \mathrm{div}(\kappa \mathrm{grad} T) + \mu J \tag{5}$$

and eliminating in (5) the pressure gives, using $c_p - c_v = R$:

$$\rho c_v \left(\frac{DT}{Dt} + (\gamma - 1) T \mathrm{div}\,\underline{u} \right) = \mathrm{div}(\kappa \mathrm{grad} T) + \mu J \tag{6}$$

This last form is important for Kirchhoff's analysis; it permits an exact split between div $\underline{u}$ and the dissipation function vs. expressions containing the temperature:

$$(\gamma - 1)\,\mathrm{div}\,\underline{u} - \frac{\mu J}{\rho c_v T} = \frac{\mathrm{div}(\kappa\,\mathrm{grad}T)}{\rho c_v T} - \frac{1}{T}\frac{DT}{Dt} \qquad (7)$$

The ultimate simplicity in the formulation of the energy equation is known to follow upon the introduction of a function of state called the entropy s, defined by the relation $T\,ds = c_p dT - dp/\rho$; s fulfills the equation

$$\rho T \frac{Ds}{Dt} = \mathrm{div}(\kappa\,\mathrm{grad}T) + \mu J \qquad (8)$$

It is noted that the neglect of the e_{ik} for $j = k$ is justified here as the effects of shear are dominant for the phenomena that are considered in this paper..

3. A first method of linearization.

We proceed to the task of harmonizing the conservation of energy with the conservation equations of mass [already included in (4)]

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \mathrm{div}\,\underline{u} = 0 \tag{9}$$

and of momentum, i.e., the Navier-Stokes equations,

$$\frac{D\underline{u}}{Dt} + \frac{1}{\rho}\,\mathrm{grad}p = \nu\nabla^2\underline{u} + \nu'\,\mathrm{graddiv}\underline{u} \tag{10}$$

where $\nu = \mu/\rho$ and ν' is Stokes' second viscosity. Here the coefficients of the viscous terms are given in their incompressible form, i.e., their possible temperature-dependence is ignored. It has been already noted in connection with the discussion of the coefficient of heat conduction that for big changes of temperarure in one spatial direction, a remedy for this simplification will be found. For the modest ranges of variation of the temperature envisaged originally by Kirchhoff, the form (10) of the equations of motion is adequate.

Here we depart from Kirchhoff's analysis and intend to find first an equation fulfilled by density fluctuations ρ', in a gas where the undisturbed state is isothermal. We first linearize density and pressure ($\rho = \rho_m + \rho'$, $p = p_m + p'$) in both the continuity and the momentum equations and then remove the vector $\underline{u}$ by cross-differentiation:

$$\frac{\partial^2\rho'}{\partial t^2} - \nabla^2 p' = (\nu + \nu')\,\nabla^2\frac{\partial\rho'}{\partial t} \tag{11}$$

This is now to be combined with the linearized version of the energy equation as given by (5), which yields with κ = const.

$$\frac{\partial p'}{\partial t} - a^2 \frac{\partial \rho'}{\partial t} = (\gamma - 1)\kappa \nabla^2 T' \qquad (12)$$

By use of the equation of state, the differential dT' in the heat conduction term is now replaced by dp' and dρ'. The result of this straightforward operation is

$$\frac{\partial \rho'}{\partial t} - \frac{1}{a^2}\frac{\partial p'}{\partial t} = \frac{\kappa}{c_p \rho_m}\nabla^2 \rho' - \frac{\kappa}{c_v \rho_m}\frac{1}{a^2}\nabla^2 p' = \frac{c_x}{c_v}\frac{\nu}{\gamma}\nabla^2 \rho' - \frac{c_x}{c_v}\frac{\nu}{a^2}\nabla^2 p' \qquad (13)$$

The final form of (13) is obtained by setting $\kappa = c_x \mu$; this expresses the fact that the same mechanism that transfers momentum in the gas is also responsible for the transfer of energy. Dimensional considerations (and Maxwell's original theory) require only that c_x *is* a specific heat.

Next the Laplacian of the pressure is eliminated between (12) and (13):

$$\frac{\partial p'}{\partial t} = \left(a^2 + \frac{c_x}{c_v}\frac{\partial}{\partial t}\right)\frac{\partial \rho'}{\partial t} - \left(\frac{c_x}{c_v}\frac{\nu}{\gamma}a^2 + \nu(\nu + \nu')\frac{\partial}{\partial t}\right)\nabla^2 \rho' = 0 \qquad (14)$$

To derive an equation for the density alone, we have first to take the Laplacian of (14). At this point, we can not avoid the occurrence of terms of the form $\nabla^2 a^2$, i.e., terms in which the Laplacian of the sound speed appears. This does not matter for isothermal problems, but shows that many new terms are generated when there is a temperature stratification in the

gas. Then it is doubtful that the presented approach can lead to useful results.

Returning to the isothermal case, we find that ρ' fulfills the equation

$$\frac{\partial^3 \rho'}{\partial t^3} - \left[a^2 + (\nu_1 + \nu_2)\frac{\partial}{\partial t}\right]\nabla^2 \frac{\partial \rho'}{\partial t} + \left[\frac{\nu_2}{\gamma} a^2 + \nu_1 \nu_2 \frac{\partial}{\partial t}\right]\nabla^4 \rho' = 0 \qquad (15)$$

Here $\nu_1 = \nu + \nu'$ and $\nu_2 = c_X \nu / c_V$. This equation is identical to one derived by Kirchhoff for the temperature disturbance T'. Here we found it satisfied by ρ'. Actually Kirchhoff's result is also correct. To prove it, we have to accept what we call (lacking better words) Kirchhoff's anticipation of the Heaviside calculus. We identify the *operator* $\partial/\partial t$ with an *algebraic quantity* $\partial/\partial t = h$ (say). Then Kirchhoff's equation is obtained from (15) upon division by h. All terms are divisible by h with the exception of one, which gives, in the notation used by Kirchhoff, the term $\nu_2 a^2 / h\gamma$.

Kirchhoff's derivation of the equation for T' uses some rather unconvincing arguments. The following derivation is proposed instead. It is noted that it will *need* Kirchhoff's Heaviside convention, while its use in (15) was "voluntary".

4. Second method of linearization

The critical term in the Navier-Stokes equations that we have to reconsider is the one that is already contained in the Eulerian equations. With the use of the entropy s this term is linearized as follows:

$$\frac{dp}{\rho} \rightarrow c_p dT' - T_m ds' = c_p dT' - \frac{T_m}{h}\frac{\partial s'}{\partial t'} = c_p dT' - \frac{1}{h\rho_m}\kappa\nabla^2 T' \qquad (16)$$

The second step makes use of the Heaviside convention and the third of the linearized entropy equation (8). Eq. (16) is to be inserted into the linearized form of the Navier-Stokes equations (10); then the operation div $\underline{u}$ leads to

$$\frac{\partial \mathrm{div}\,\underline{u}}{\partial t} + c_p\nabla^2 T' - \frac{c_x \nu}{h}\nabla^4 T' - (\nu + \nu')\nabla^2 \mathrm{div}\,\underline{u} = 0 \qquad (17)$$

This is combined with the linearized form of the energy equation (7)

$$\mathrm{div}\,\underline{u} = \frac{1}{a^2}\left(\gamma c_x \nu \nabla^2 T' - c_p \frac{\partial T'}{\partial t}\right) \qquad (18)$$

Finally, div $\underline{u}$ is eliminated between (17) and (18) to give

$$h^2 T' - \left(a^2 + h[(c_x / c_v)\nu + \nu + \nu']\right)\nabla^2 T' + \frac{c_x}{hc_p}\nu[a^2 + h\gamma(\nu + \nu')]\nabla^4 T' = 0 \qquad (19)$$

Thus Kirchhoff's statement is proven. Eq. (19) for T' might be valid even in a gas where T_m is not constant, as the derivation did not require to take the Laplacian of T_m.

5. The limits of Kirchhoff linearization.

At this point, one may ask whether p' too fulfills the same equation. It does, but the proof based on the results obtained thus far turns out to be unexpectedly difficult. It involves manipulation of the equations until an expression emerges that is divisible by $\nu_2 - \nu_1$. Upon this division, the expected equation for p' is obtained.

Actually, no significant general conclusions can be deduced from the results obtained thus far as all three equations are inhomogeneous, i.e., do not admit arbitrary amplitudes. This is seen for the limit $\partial/\partial t = 0$ ($h = 0$). (We have to multiply Eq. (19) for T' by h before considering this limit.) In all cases, there is an unbalanced "secular term" left that could grow indefinitely with time t:

$$\frac{\nu_2}{\gamma} a^2 \nabla^4 \rho' \cdot t = ?$$

Kirchhoff intends to find a homogeneous equation by eliminating the inhomogeneous term. He actually pursues this program only for the equation fulfilled by the density fluctuations. To follow Kirchhoff's analysis in detail is very difficult. However, the essence of the problem can be seen by treating a specific case: the determination of the particle velocity u in the direction of the sound propagation. Then, the equation that has to be fulfilled *together* with (15) is the linearized continuity equation

$$-h\frac{\rho'}{\rho_m} = \frac{\partial u}{\partial x} \qquad (20)$$

The combined equations (15) and (20) determine u. We can not write down an explicit equation for u because we can not invert (15), but we can argue that with the degree of freedom given by the "free function of integration" implied by (20), the desired solution must exist. This is true and we have a prominent example of successful Kirchhoff linearization.

Nevertheless, here at the final proof of its success, we propose to "abandon" Kirchhoff linearization or rather to try to construct alternatives. We are naturally worried by the fact that no way is evident to come to a homogeneous equation for the other state variables. But we intend to press the point that even the possible common solutions of (15) and (20) which must exist do not lead to modes of overriding physical importance. They will be overpowered by modes in which damping, both due to friction and to heat conduction, is much more concentrated in wall regions.

What we maintain is that the Kirchhoff solutions are not "truly" homogeneous inasmuch as they can not be renormalized freely. They must belong to the subgroup that is subject to the restriction (20). This subgroup of solutions will be shown to be physically uninteresting, essentially by the construction of an alternative to the Kirchhoff linearization which will lead to a group which is simpler and will permit physical applications free of the "crippling" restriction that it must fulfill (20). It will have its own inherent limitations but will cover a wider range of phenomena than the Kirchhoff group.

The only quantity for which a homogeneous solution has emerged by using Kirchhoff's linearization is the particle velocity, and it is known that for this quantity a non-linear equation is needed for a complete physically

valid description. The amplitude in the linear Kirchhoff group is restricted by direct microscopic molecular-kinetic mechanisms.

Finally we recall the great success of Kirchhoff linearization in treating connections to free molecular flow, which was explored in the theory of Greenspan [6]. Our interest lies in the opposite extreme. Historically the Kirchhoff theory was subjected to much scrutiny but it seems few radical changes have gained acceptance.

The following discussion is restricted to the treatment of the basic longitudinal mode. Kirchhoff's original linearization scheme permits a generalization to transverse modes which we can not hope to achieve with the proposed new approach without efforts that go far beyond the present state of our understanding of these problems. The riches in the possible transverse modes that are not yet fully explored are truly overwhelming.

6. The "third way"

Hoping for a more radical way to achieve progress, we look at the linearized momentum equation which is to be targeted for inclusion. We are also ready to forget about Stokes' second viscosity and for the time being to ignore the temperature-dependence of the first. Then the linearized x-component of the Navier-Stokes equations gives (21)

$$\frac{1}{\rho_m}\frac{\partial p'}{\partial x} + \frac{\partial u}{\partial t} = \frac{\mu}{\rho_m}\nabla^2 u \qquad (21)$$

It is only natural to look at the juxtaposition with the energy equation in the following form, which can be easily derived by rearranging (12):

$$\frac{1}{\gamma p_m}\frac{\partial p'}{\partial t} + \operatorname{div} \underline{u} = \frac{\kappa}{\rho_m c_p}\nabla^2\left(\frac{T'}{T_m}\right) \qquad (22)$$

We see that if we ignore heat conduction in (22) and viscosity in (21), then we get the classical equations for sound propagation. There are two important facts that form obstacles for a direct use of this observation for an acoustic theory. First: we can aim for an equation for p' *alone* or for u *alone*, but we don't know which is the *right* way. Second: (22) contains div $\underline{u}$ and not just $\partial u/\partial x$. We face this second difficulty first.

To ignore this fact is widespread in acoustics practice and even in texts, but we can not expect to establish a theory that can compete with Kirchhoff if we accept such an unjustifiable step. We propose a way out which first means that we have to lower our expectations on what a linear

theory can give, and leave it for later to try to regain lost ground.

An idea for proceeding is suggested by Prandtl's boundary layer theory which leads to the assumption that pressure changes perpendicular to the direction of sound propagation can be neglected:

$$\frac{\partial p'}{\partial y} = \frac{\partial p'}{\partial z} = 0 \qquad (23)$$

This was "assumption number one" in my 1969 paper [3]. I proceeded to introduce this assumption into the heat conduction term, which then reduced formally to a density diffusion term. My experience has shown that this step was generally received with strong suspicions, and while the following analysis will support in the end the validity of this radical idea, it is far more convincing to avoid this mental jump and look first for assumptions which can be better defended.

The second simplification proposed is to neglect in the Laplacian the derivative in the direction parallel to the wall surface. The consequences of this assumption are spectacular, even though the term affected is already linear. In somewhat loose terms, this idea can be explained as follows. The development of the flow near the wall, which according to the Navier-Stokes equations is given as an "elliptic" problem, becomes "parabolic" by this simplification. Prandtl has shown [7] that this step is asymptotically correct for thin boundary layers in the limit of high Reynolds numbers.

The ambiguity in the expression "parallel to the wall" can be removed by considering only wall surfaces with generators parallel to *one* axis. This is now accepted for all following steps in the analysis. Then, Prandtl's second simplifying assumption means a change in the definition of

the Laplacian:

$$\nabla^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{24}$$

The neglected term in (24) is the only one that was kept in the previous Section for the investigation of particle motion in the direction of the sound propagation. The consequence of (24) is a disengagement of the flow problems parallel and across the direction of sound propagation.

With these assumptions, the problem remains non-trivial even if we restrict our attention to the *averaged* equations (21) and (22). Possible shortcomings of the analysis which could be caused by the wrong treatment of the div $\underline{u}$ term are avoided. We apply to our equations the operator $\iint\ldots\,(dA/A)$, where A is the cross-sectional area. Assumption (24) leads to $\iint p'(dA/A) = p'$. We have to introduce a new notion by defining the volume flux φ as follows:

$$\frac{1}{A}\iint u\,dy\,dz = \varphi \tag{25}$$

The application of the averaging operator to the Laplacians leads to the introduction of wall values, by Gauss' theorem. Their evaluation with (24) and with the homogeneous boundary conditions $u = T' = 0$ on the circumference is a classical problem for which analytic solutions are known for many shapes. What is naturally not known is how the shapes differ in their behavior in response to the fact that $\partial^2 u/\partial x^2$ has been neglected.

For the classical case where (24) has sufficient symmetries for an elementary evaluation, explicit results are given. The two possible cases are

covered by using the Laplacian in the form

$$\nabla^2 = \frac{\partial}{r^j \partial r}\left(r^j \frac{\partial}{\partial r} \right) \tag{26}$$

where r is the distance from the origin, and j = 0 or 1, depending whether the problem is plane or axisymmetric. Integrating (22) with the use of (26) gives

$$\frac{1}{\gamma p_m}\frac{\partial p'}{\partial t} + \frac{\partial \varphi}{\partial x} = \frac{(1+j)\kappa}{r_w \rho_m c_p T_m}\frac{\partial T'}{\partial r}\bigg|_w \tag{27}$$

The same operations applied to the momentum equation (22) lead to

$$\frac{1}{\rho_m}\frac{\partial p'}{\partial x} + \frac{\partial \varphi}{\partial t} = \frac{(1+j)\mu}{r_w \rho_m}\frac{\partial u}{\partial r}\bigg|_w \tag{28}$$

It is noted that $r_w/(1+j)$ is known as the "hydraulic radius" r_H.

As the fluctuations of the state variables in any transverse plane are in phase, we have from the equation of state

$$\frac{1}{T_m}\frac{\partial T'}{\partial r}\bigg|_w = -\frac{1}{\rho_m}\frac{\partial \rho'}{\partial r}\bigg|_w \tag{29}$$

This changes (27) into the form given in [3] where the temperature fluctuations were replaced by density fluctuations. However, the use of the density was an unnecessary complication, as boundary conditions are given in terms of T' (and of u). Thus the criticism of [3] was also justifiable.

Looking back we can see that the essential point of departure between the two approaches is found in the formulation of the effect how energy fluctuations connect to motion. In Kirchhoff's theory, temperature

fluctuations are "caused" by density fluctuations, while in the present approach, temperature changes are caused by pressure changes. The point of departure can be seen by looking at (11), the equation that can *not* be fulfilled in the new approach. This is just as well, as (11) introduces an unbalanced pressure term in the steady state which is the source of the difficulties encountered in establishing a linearization valid for all functions of state. On the other hand, the new approach has led us to the pair of equations (27)-(28) but without instructions how to harmonize their conflicting statements.

At this point we realize that had we restricted our attention to a hypothetical gas with no heat conduction, we would have no conflict. Actually we would be finished with our analysis, as the averaging process leads to the desired solution. We obtain equations for p' and φ alone by cross-differentiation from (27) and (28). The equation for p' is (with a simple and obvious change of notation)

$$\frac{\partial^2 p'}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 p'}{\partial t^2} = \frac{1}{r_H}\frac{\partial \tau_W}{\partial x} \qquad (30)$$

We can easily argue that (i) it is more important to fulfill (30) than to fulfill the equation for φ (not shown), and (ii) that the value of φ is already known as it is included in the solution of (30). It is found upon division of the relation (28) by h (i.e., integration with respect to time).

The important point is that we recognize the necessity of solving the equation for p' *first*. Then we deduce the attendant flux that follows after the equation for p' has been fulfilled. Thus, a "hierarchy" exists in the order

in which the basic equations must be solved.

The upshot is that analysis which includes the effects of heat conduction *begins* and *ends* with the solution of the following equation:

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho_m}\frac{\partial p'}{\partial x}\right) - \frac{1}{\gamma p_m}\frac{\partial^2 p'}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{\nu_1}{r_H}\frac{\partial u}{\partial r}\bigg|_w\right) - \frac{\nu_2}{r_H}\frac{\partial}{\partial t}\left(\frac{\partial \theta}{\partial r}\bigg|_w\right) \qquad (31)$$

Note: ν_1 and ν_2 were defined in (20). The abbreviations $\theta = T'/T_m$. and r_H are introduced for convenience. The form of (31) acknowledges that ρ_m (in contrast to p_m) can be a function of x.

The fact that to fulfill (31) is necessary and sufficient for the solution of the problem with heat conduction emerges as the main result of the new linearization. At this point the continuation of the analysis without the possibility to discuss specifics becomes too cumbersome. The next step is the evaluation of the wall values to be introduced in (31). For this we have to return to the expressions for u and T' before averaging.

7. Profiles. The isothermal solution

The evaluations of this Section are based on the idea that in any cross-section x = const., both u and T' fulfill locally linear inhomogeneous equations, whereby the forcing terms are given by p' in (31). The homogeneous parts are solutions of Laplace's equation, which is used in the simplified form (24) for the actual evaluations. Unavoidable for a concise treatment of this problem is to take the step from Kirchhoff's anticipated Heaviside calculus to modern Heaviside calculus, where we set $\partial/\partial t = i\omega$. This step has become so routine that its problems are largely forgotten, but we will have to face them later. (To set $\partial/\partial t = -i\omega$ does not help.)

The velocity profile follows from (21). The expression for u which solves (21) locally, i.e., with x as a parameter, and is adjusted to the no-slip condition at the wall is:

$$u = \frac{i}{\omega\rho_m}\frac{dp'}{dx}\left\{1 - \frac{F(\eta)}{F(\eta_w)}\right\} \tag{32}$$

where F is a function of a new "dynamic similarity variable" η defined as

$$\eta = \left(\frac{i\omega}{\nu_1}\right)^{1/2} r \tag{33}$$

in which F fulfills the homogeneous equation

$$i\omega F^{(j)} = \nu_1 \nabla^2 F^{(j)} = \nu_1 \frac{\partial}{r^j \partial r}\left(r^j \frac{\partial F^{(j)}}{\partial r}\right) \tag{34}$$

where we note the cases j = 0 and j = 1:

$$F^{(0)} = \cosh \eta\,, \qquad F^{(1)} = J_0(i\eta) = I_0(\eta) \tag{35}$$

The explicit use of the superscripts will be suppressed whenever possible.

The wall velocity gradient follows directly from (32) and is, upon adjusting phases, given by

$$\left.\frac{\partial u}{\partial r}\right|_w = \frac{1}{\sqrt{i\omega\rho_m\mu}} \frac{dp}{dx} \frac{F'(\eta_w)}{F(\eta_w)} \tag{36}$$

This classical result shows the well-known phase difference between wall shear and pressure gradient.

For the term to be inserted in (31) we find from (36), using (33):

$$\frac{\nu_1}{r_H} \left.\frac{\partial u}{\partial r}\right|_w = \frac{(1+j)F'(\eta_w)}{\eta_w F(\eta_w)} \frac{1}{\rho_m} \frac{\partial p'}{\partial x} \tag{37}$$

We introduce a "complex coefficient of friction" f defined as

$$f = \frac{(1+j)F'(\eta_w)}{\eta_w F(\eta_w)} \tag{38}$$

which is a function of the complex similarity parameter η_w, also called here the "complex Stokes parameter"

$$\eta_w = \left(\frac{i\omega}{\nu_1}\right)^{1/2} r_w \tag{39}$$

We will use $f = f(\eta_w)$ as *the* parameter in our equations, a choice that will bring substantial advantages. We note that a complex number carries extra information if we know the rules how to use its "phase".

To find the temperature distribution in the gas, the energy equation is invoked, but we have to use a form in which the pressure is the only non-thermal variable. The equation before linearization that fulfills this requirement is (5). Linearized it gives

$$\frac{\partial\theta}{\partial t} - \frac{\gamma - 1}{\gamma p_m}\frac{\partial p'}{\partial t} = \nu_2 \nabla^2\theta \tag{40}$$

For a sufficiently high heat capacity (and reasonably high conductivity) of the isothermal duct, the boundary condition at the wall is $\theta_w = 0$. The solution of (40) adjusted to this condition is

$$\theta = \frac{(\gamma - 1)p'}{\gamma p_m}\left\{1 - \frac{F^*(\eta)}{F^*(\eta_w)}\right\} \tag{41}$$

where F^* is a solution, in terms of a newly defined similarity variable

$$\eta^* = \left(\frac{i\omega}{\nu_2}\right)^{1/2} r \tag{42}$$

of the homogeneous equation

$$i\omega F^* = \nu_2 \nabla^2 F^* \tag{43}$$

We can use the same functionals in the equations based on the use of ν_1 and on ν_2 by introducing the Prandtl number σ defined as

$$\sigma = \frac{\nu_1}{\nu_2} = \frac{\mu c_p}{\kappa} \tag{44}$$

Comparison between (34) and (43) shows that

$$F^*(\eta) = F(\eta\sqrt{\sigma}) \tag{45}$$

The term in (31) which contains the wall temperature gradient follows from (40) and is now given by

$$\frac{\nu_2}{r_H}\frac{\partial\theta}{\partial r}\bigg|_w = -\frac{(1+j)F'(\eta_w\sqrt{\sigma})}{\eta_w\sqrt{\sigma}F(\eta_w\sqrt{\sigma})}\frac{(\gamma-1)i\omega p'}{\gamma p_m} \tag{46}$$

It is seen that if we define in (46) a coefficient f* as in (47) below, the relation between f and f* follows the rule (44):

$$f^* = \frac{(1+j)F'(\eta_w\sqrt{\sigma})}{\eta_w\sqrt{\sigma}\,F(\eta_w\sqrt{\sigma})} = f(\eta\sqrt{\sigma}) \tag{47}$$

The final step is to join the right-hand side terms in (31), given by (38) and (46),with the corresponding "inviscid" terms on the left:

$$\frac{\partial}{\partial x}\left(\frac{1-f}{\rho_m}\frac{\partial p'}{\partial x}\right) + \frac{1+(\gamma-1)f^*}{\gamma p_m}\omega^2 p' = 0 \tag{48}$$

In the isothermal case, which we discuss first, all complex quantities are independent of x. In the case without heat conduction we have $f^* = 0$ and we recognize (for constant density) classical results. To obtain solutions which represent damped waves, we assume that the frequency has the form $\omega = \omega_0\sqrt{1-f}$ with ω real, so that (48) becomes a real equation and the imaginary part of ω leads (following well-known rules involving the choice of the proper branch of the root) to a solution for a damped wave.

In the case with heat conduction, the second term in (48) is also

complex. However, if we divide the homogeneous equation (48) by the factor $1 + (\gamma - 1) f^*$, we have reduced the problem to the former case.

We could go a step further and use the fact that the absolute values of f and f^* are small compared to 1, so that a new coefficient $1 - f - (\gamma - 1) f^*$ in the first term of (48) should be sufficiently accurate to represent both "loss" terms. Experiences accumulated in decades strongly discourage this step, even though f and f^* are both small. However, their *position* in (48) *has* physical significance. The friction factor f originates obviously in effects of the wall shear, while f^* is "caused" by fluctuations of the boundary layer thickness for which the expression "surface pumping" became generally accepted.

We also realize that small friction factors correspond to thin boundary layers, and that in the sense of Prandtl a limit on the validity of the present analysis has to be expected, as the effects of viscosity and heat conduction spread through the whole cross-section. However, as we have found analytic expressions for both f and f^*, no clear criteria emerged that could be used to define limits of validity.

We wish to look at the positive side of this statement and to emphasize that we have obtained analytic continuations of thin boundary-layer results, and we are ready to use them with the *proper caution.*

The question is: what normalization has to be applied to (48) if we require that the term containing $\partial^2 p'/\partial t^2$ represents the physical effects of damping? The answer is that we can multiply (48) by any number which makes (or leaves) the coefficient of $\partial^2 p'/\partial t^2$ *real.* Thus, if heat transfer is neglected, (48) is the final answer.

In the general case, we have to make the wave number "apparently" complex so that the frequency can become "truly" complex. Thus, if we introduce the complex wave number k defined by

$$k^2 = \frac{\omega^2}{a^2}\frac{1+(\gamma-1)f^*}{1-f} \tag{49}$$

then in the isothermal heat conducting case the problem is reduced to the solution of the equation

$$\frac{1}{k^2}\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial t^2} = 0 \tag{50}$$

and the responsibility to find the proper branch which represents the physically realizable solution is shifted to the task of finding the proper wave number.

The option to use a wave number instead of a wavelength is a trivial one, but the term which *must* become real by proper normalization is, in Kirchhoff's notation, h^2p'.

8. The "complex Kirchhoff parameter"

Our plan to use f as the complex analytic parameter faces a certain difficulty as we have to deal with *two* complex expressions, f and f*, and their connection is non-analytic. This is an inherent complication of the Kirchhoff theory and all its variants. We have to deal with two systems, connected by the affine transformation (45), and the results have to be fitted into the same "channel width". The "complex Kirchoff parameter" (42), the counterpart of (39), fixes the scale of heat conduction effects. The ratio of the scales is given by the square root of the Prandtl number, as shown in (44)-(45).

The fact that the Prandtl number of most simple gases is near 1 hides the importance of the necessity of dealing with two scales. The consequence in practice is that it suffices to use only the one similarity parameter defined in (39). We find that the complex ratio F'/F in both (38) and (47) tends to 1 fast enough (for big arguments, which means sufficiently thin boundary layers) so that we can use $f^* = f/\sqrt{\sigma}$ as a good approximation for much of the useful range of f.

This approximation, which is in wide use since the times of Kirchhoff, retains an interesting property of the general flow problem: it diverges for vanishing Prandtl numbers. This is not the first time that this result is noted but maybe this time around an idea will emerge how to deal with the unresolved problem $\sigma \to 0$. There is a marked difference in boundary layer behaviors for high and low heat conductivity. For low

conductivity no dramatic effects can be expected, as the thermal boundary layer is completely inside the viscous layer. If, on the contrary, conductivity is high enough so that heat effects are felt outside the viscous layer, then the properties of that part of the thermal layer seem to be insufficiently described by the energy equation alone. In a linearized theory, terms needed for a complete momentum balance are missing and terms due to density fluctuations are not balanced. It can be conjectured that this scenario leads to a "shortcut to turbulence".

As noted before, our analysis provides a built-in continuation to layers that fill a full cross-section. Here we emphasize again the need for caution when we interpret these results in tubes that are too narrow.

The question is brought into focus by considering the well-known property of Rayleigh acoustic streaming, namely, that it is independent of the coefficient of viscosity. It is possible to imagine that sufficiently thin thermal layers do not interfere with this conclusion, and observational evidence supports this idea. Actually, Rayleigh's theory apparently covers successfully also the cases where the Prandtl number is less than 1 but within the limits known to exist for simple gases. However, there must be a limit for high conductivity where linearization fails completely, for all isothermal problems.

9 The flux.

We now intend to implement the simplifications that follow from the "useful but limited" approximation $f^* = f/\sqrt{\sigma}$ which permits the use of analytic approximations for all coefficients. We already have agreed to solve for the pressure first. Now we wish to find the attendant values of the volume flux. Here we have a certain freedom in the normalization of this quantity as long as it is assured that it fulfills a homogeneous equation. To keep things simple, we state that we do not want to introduce any new differentiations to those which were already needed to obtain (48). In connection with the more general result (31) we already noted that ρ_m can be a function of x, in contrast to p_m. These considerations suggests the use of a new variable q', a "substitute flux", defined as follows:

$$q' = \frac{(1-f)a^2}{\omega^2}\frac{dp'}{dx} \tag{51}$$

For inviscid flow, q' reduces to $\gamma p_m(i/\omega)\varphi$, as seen from the momentum equation (28), which gives φ (upon division by h). We solve (48) for p'. At this point, however, we consider it as very important that the emerging equation be analytic in f. Thus we write

$$p' = -\frac{1}{1+cf}\frac{dq'}{dx} \tag{52}$$

where

$$c = \frac{\gamma - 1}{\sqrt{\sigma}} \tag{53}$$

gives in (52) the close approximation to f^*.

The pair of equations (51)-(52) contains the same amount of information as (49)-(50). We are therefore permitted - and actually urged - to forget (49)-(50), particularly also the definition of the wave number (49). It is contained in the pair (51)-(52), together with the relation among the end-conditions for both variables which permit the solution of the homogeneous problem in either variable.

The analytic form of the wave number associated with these results is

$$k = \frac{\omega}{a}\left(\frac{1+cf}{1-f}\right)^{1/2} \tag{54}$$

but we insist that vital information would be lost if we simplify further, at this stage, for small f.

10. Effects of temperature stratification

Proceeding to the case of a long and sufficiently narrow tube of constant cross-section but with temperature stratification along its axis, we are prepared to use different approximations for heat conduction in the gas across and along the tube. First however we have to face again the problem of linearization. Here we can achieve a small but welcome simplification by considering the linearization of the energy equation directly, without introducing first the linearized continuity equation. Then we do not have to repeat the averaging process, which is already built-in in all our results. Thus the dominant effect of the stratification will be expressed by replacing T' by $T' + u\, dT_m/dx$. The same rule has to be applied to the density (if needed), while the constancy of p_m means that p' is left unaffected. Thus the stratification of all functions of state can be expressed by a single parameter

$$d\log T_m/dx = -d\log \rho_m/dx \equiv \Theta \qquad (55)$$

In the strongly simplified analysis that follows, only effects directly associated with this parameter will be considered

The fundamental energy equation (5), which was already used in its isothermal and linearized form (40), is now rewritten to account for the stratification effect expressed by (55) and becomes

$$\frac{\partial \theta}{\partial t} + \Theta u - \frac{\gamma - 1}{\gamma p_m}\frac{\partial p'}{\partial t} = \nu_2 \nabla^2 \theta \qquad (56)$$

The simplicity of this equation is deceptive. We recall that $\theta = T'/T_m$; thus, if we apply (56) over a wide range of T_m, this equation is not sufficiently accurate in the actual small disturbance variable, which is T'. With our willingness to use a coefficient ν_2 which is a function of x we can postpone the implementation of a remedy. Of immediate concern is the effect of the u-term. In linear approximation we introduce a coefficient proportional to the temperature gradient Θ, which is a given function of x.

In my paper of 1969, I have proposed a step that has led to limited success after a long detour. The presentation here is aimed to better define the limits and to provide shortcuts.

We return to the isothermal case covered by (40) and recall that this equation was used for a local solution at a given position x, with the pressure p' acting as the forcing term. The same interpretation applied to (56), with the use of (32) for u, requires the solution of the equation

$$i\omega\theta - \nu_2\nabla^2\theta = \frac{i\omega(\gamma-1)p'}{\gamma p_m} - \frac{i\Theta}{\omega\rho_m}\frac{dp'}{dx}\left\{1 - \frac{F(\eta)}{F(\eta_w)}\right\} \qquad (57)$$

and it is seen from (32) and (40) that for $F = F^*$, $\sigma = 1$, the solution of (57) becomes singular.

Instead of restricting our attention to the special case where the Prandtl number equals 1, we solve (57) for arbitrary values of σ and go to the limit $\sigma = 1$ as needed. The solution of (57) is then

$$\theta = \left(\frac{\gamma - 1}{\gamma} \frac{p'}{p_m} - \frac{\Theta}{(1-\sigma)\omega^2 \rho_m} \frac{dp'}{dx} \right) \left\{ 1 - \frac{F^*(\eta)}{F^*(\eta_w)} \right\} + \frac{\sigma \Theta}{(1-\sigma)\omega^2 \rho_m} \frac{dp'}{dx} \left\{ 1 - \frac{F(\eta)}{F(\eta_w)} \right\} \qquad (58)$$

The procedures that led to the determination of wall values, which resulted in the relations (45) and (48), are now repeated for the evaluation of wall values in (58). The result is

$$\frac{d}{dx} \left(\frac{1-f}{\rho_m} \frac{dp'}{dx} \right) - \Theta \frac{f^* - f}{1-\sigma} \frac{dp'}{dx} + \frac{[1 + (\gamma - 1) f^*] \omega^2 p'}{\gamma p_m} = 0 \qquad (59)$$

There is a new term proportional to dp'/dx, with a coefficient depending on the complex Stokes parameter (39), which signifies damping or excitation. Note that $1/\rho_m$ is differentiated in the first term of (59), and this still represents the most basic effect of stratification. With a variable sound speed along x, the result (59) is physically meaningless applying traditional interpretation.

The way out is to live with variable sound speed until a re-interpretation with piecewise constant temperature becomes available. This solution was indicated in the Introduction and is not without precedent.

Next we take advantage of a possibility to introduce the temperature dependence of the viscosity, by a change in the coefficients in (59). It is proposed to follow Kramers and to use a simple power-law dependence of μ on the absolute temperature :

$$\mu = \text{const.}(T_m)^{\beta} \tag{60}$$

This law can be "absorbed" into a purely formal re-definition of the coefficient of dp'/dx in the new second term in (58) as follows. We are looking for a new complex function $E(\eta_w)$ (say) such that p' fulfills the following new equation:

$$\frac{1}{E}\frac{d}{dx}\left(E\frac{1-f}{\rho_m}\frac{dp'}{dx}\right) + \frac{[1+(\gamma-1)f^*]\omega^2 p'}{\gamma p_m} = 0 \tag{61}$$

which has only two terms. Comparing (61) and (59) shows that if the new second term in (59) fulfills the relation

$$\frac{1}{E}\frac{dE}{dx}\cdot\frac{1-f}{\rho_m}\frac{dp'}{dx} - \Theta\frac{f^*-f}{1-\sigma}\frac{dp'}{dx} = 0 \tag{62}$$

then indeed the form (62) emerges. This relation is divisible by dp'/dx and depends only on the coefficients in (59). Integrated, (62) gives with the use of (55)

$$\frac{d\log E}{dx} = -\frac{f^*-f}{(1-\sigma)(1-f)}\frac{d\log T_m}{dx} \tag{63}$$

This result can now be combined with the power-law dependencies of η_w and μ, given by (39) and (60). The logarithmic x-dependence of these terms fulfills the relation

$$d\log\eta_w = \frac{1}{2}(d\log\rho_m - d\log\mu) = -\frac{1+\beta}{2}d\log T_m \tag{64}$$

and the final formula for E is

$$E = \exp\left(\frac{2}{1+\beta} \int \frac{f^* - f}{(1-\sigma)(1-f)} \frac{d\eta_w}{\eta_w} \right) \tag{65}$$

The "integrating factor" E is now defined by σ and β, up to a multiplicative constant. Inviscid flow is found in the limit of the Stokes parameter tending to infinity, and it is often convenient to set E = 1 in this limit. The function defined by (65) can be evaluated "once and for all" for any simple gas. It was put to wide use at the ETH Zurich in the years 1969 to 1985.

As already noted in 1969, the function E can be evaluated exactly for $\sigma = 1$:

$$E(\sigma = 1) = (1-f)^{1/(1+\beta)} \tag{66}$$

In preparing the present paper, the simplification $f^* = f/\sqrt{\sigma}$ was already introduced in Section 9. The same step, applied to (65), gives

$$E = (1-f)^m \tag{67}$$

where

$$m = \frac{2}{(1+\beta)(\sigma+\sqrt{\sigma})} \tag{68}$$

The use of the E-function suggested by (61) is now to set

$$q' = \frac{(1-f)Ea^2}{\omega^2} \frac{dp'}{dx} = \frac{(1-f)^{1+m} a^2}{\omega^2} \frac{dp'}{dx} \tag{69}$$

which is to be complemented by

$$p' = -\frac{1}{(1+cf)E}\frac{dq'}{dx} = -\frac{1}{(1+cf)(1-f)^m}\frac{dq'}{dx} \qquad (70)$$

Thus the structure of the basic equations (51)-(52) remained unaffected, but the results have one fatal flaw: an x-dependent speed of sound.

Before proceeding, we have to answer the question: are our variables continuous? For the pressure p', we demand that it be continuous for the family of solutions to be considered here. It is known that near resonance, discontinuous solutions exist; they were treated in a seminal paper by Chester [8] in 1964. His study confirms that if there is a discontinuity, it has to move relative to the tube, with (very nearly) the speed of sound. These solutions are not homogeneous; Chester gave the first complete explanation of the driving mechanism. His theory includes the effects of wall friction, and a complete thermal balance including the effect of heating by weak shocks is possible [9].

The theme of our study is the quest for homogeneous solutions, and we leave the consideration of inhomogeneous solutions to a few remarks at the end of the paper.

Requiring continuous p also leads to a continuous q, by noting the relationship of this quantity to the definition of the flux (51).

11. The discontinuous temperature model 1: general discussion.

We are now ready, following Kramers [3], to consider a gas column divided into two sections that have different temperatures T_{m1} and T_{m2}. Then it is physically clear that a simple fully periodic small disturbance solution can *not* exist. "Simply" means periodic upon N wave passages along the tube, with N of order 1. To judge this, let us think of the inviscid limit which is always included in our analysis for $f = 0$. Without *exactly* equal frequencies in the different sections, there is diffraction and reflection at the interface of the two regions, and even simple frequency ratios lead to chaotic confusion of wave patterns by the inevitable effects of the interface.

All these considerations are of interest only for free oscillations. For forced oscillations, the properties of the inhomogeneous term are dominating the resulting effects. The "frequency response" (which may be chaotic for certain regions) is the ultimate result of any inhomogeneous linear theory.

For free oscillations, however, a fact that saves the applicability of the results was recognized by Kramers [2]. We are considering cases where the ratio of the absolute temperatures between the hot and the cold parts of the gas is high. In the limit of extremely high temperature ratios, the time required for a wave to cross the hot part becomes negligibly small compared to the time "spent" in the cold part. This means that the hot part reacts "almost" like an unstructured gas-spring.

The phenomena then become very similar to those observed in the so-

called "Sondhauss" tube, which were explained in simple terms by Rayleigh [10]. A tube with one end closed and hot and the other end open and cold is considered to be a "degenerate" Sondhauss tube (or "Taconis" tube) when its cross-sectional area is constant. A "genuine" Sondhauss tube has at its hot end a bulb which is essentially spherical. The details of its shape have negligible effect on the phenomena, as long as the hot part extends to the "neck" where it joins the cold tube with constant diameter. This fact was observed by Sondhauss and explained by Rayleigh. However, the coincidence of the jump in cross-section *and* in temperature facilitates the solution of the problem, as will be seen later.

In our case, the position of the temperature jump along the tube of constant cross-section and the attendant critical temperature ratio for the onset of thermoacoustic oscillations are the *two* connected unknowns of the problem. Historically, the observation of cryogenic oscillations in helium ducts led Taconis *et al* [11] to propose a model which then was investigated theoretically by Kramers [2]. In the Taconis tube, oscillations are caused solely by "surface pumping", and the subdivision of the tube effects into "mostly spring" and "mostly mass" becomes part of the problem.

In contrast, for the original Sondhauss tube design where the bulb is only spring and the gas in the tube is only mass, the determination of the frequency following Rayleigh is sufficiently accurate. Then it will be shown that the use of the E-function will give directly the distinction between damping and excitation. Thus the constant cross-section case treated here in detail represents the most difficult case to be solved.

12. The discontinuous temperature model 2: applications

The way to use our results is to consider in each section with constant temperature a "local" or "regional" wave number k and to construct solutions for p and q that fulfill the continuity conditions that were discussed previously. Also needed are the end conditions; the case of one open and one closed end is considered first.

Let the open end of the tube be situated at $x = 0$ and the temperature be maintained at a constant value T_{m1}. At $x = \ell$ the temperature jumps to T_{m2}, and the closed end is at $x = L$. Then the continuous pressure distribution is, making use of local solutions based on the wave number defined by (54):

$$\begin{aligned} 0 \le x \le \ell: \quad & p_1 = p(\ell)\frac{\sin k_1 x}{\sin k_1 \ell} \\ \ell \le x \le L: \quad & p_2 = p(\ell)\frac{\cos k_2 (L - x)}{\cos k_2 (L - \ell)} \end{aligned} \tag{71}$$

where we use the wave numbers (54)

$$k_1 = \frac{\omega}{a_1}\left[\frac{1 + c f_1}{1 - f_1}\right]^{1/2} \qquad k_2 = \frac{\omega}{a_2}\left[\frac{1 + c f_2}{1 - f_2}\right]^{1/2} \tag{72}$$

The condition $q_1 = q_2$ that was established as "descendant" of the continuity requirement for the "flux" gives

$$\frac{[1+cf_1]E_1}{k_1}\cot k_1\ell = \frac{[1+cf_2]E_2}{k_2}\tan k_2(L-\ell) \qquad (73)$$

The case with two closed ends was treated by Ulrich Müller in his dissertation [12]. The change to the boundary conditions dp/dx = 0 at x = 0 and x = L are taken into account in the following expressions for the pressure:

$$\begin{aligned} 0 \le x \le \ell: \quad & p_1 = p(\ell)\frac{\cos k_1 x}{\cos k_1 \ell} \\ \ell \le x \le L: \quad & p_2 = p(\ell)\frac{\cos k_2 (L-x)}{\cos k_2 (L-\ell)} \end{aligned} \qquad (74)$$

The condition $q_1 = q_2$ leads to

$$-\frac{[1+cf_1]E_1}{k_1}\tan k_1\ell = \frac{[1+cf_2]E_2}{k_2}\tan k_2(L-\ell) \qquad (75)$$

In this case, the minimum temperature ratio for the occurrence of oscillations was found by Müller to be much higher than for tubes with one open end (by a factor of about 2 in helium).

The case with one open end can ideally be extended using proper symmetries to cover the case of a tube of length 2L, with two closed ends. Practically, however, the two cases differ because the open end boundary condition dp/dx = 0 is not perfectly realized when the gas in the tube exhausts into an "atmosphere" of a gas at rest. This problem was tackled by van Wijngaarden [13] [14]; to use his results for the case at hand is an interesting challenge but will not be pursued here any further.

Joining two tubes with open ends at their closed ends and then removing the solid interface leads to a configuration which obviously does not oscillate, but interesting cases can be found for configurations involving gas-liquid oscillations. These were discussed in a paper by Müller and Rott [15] but will not be considered here any further.

Actually, in the sense of Müller, we are finished with the theory, as the problem is reduced to solving (72) or (75) by search for cases where the equations are solved for real values of the frequency ω. These then represent the stability limits we were intent to find.

This, however, was not the way I proceeded in 1969, as following tradition, I inspected the results first assuming that boundary layers are very thin everywhere. At this point I hit the same barrier that stopped Kramers, who was prevented by a "quirk of nature" from enjoying the fruits of his labor. The leading order effect *for thin boundary layers* that enables the system to oscillate in helium is a non-vanishing *imaginary* part of the expression $(1+cf)(1-f)^m$ which in leading approximation is proportional to $m - c = d$ (say). We called this coefficient the "Kramers constant":

$$d = \frac{2}{(1+\beta)(\sigma+\sqrt{\sigma})} - \frac{\gamma-1}{\sqrt{\sigma}} \qquad (76)$$

For helium, $\gamma = 5/3$, $\sigma = 2/3$ (for Maxwellian molecules), and d would vanish for $\beta = .652$. The best observed value of β for a wide temperature range is .647. Whether "quirk of nature" is a legitimate notion remains an open question.

The way out I proposed was to go to second order boundary layer theory, and the implementation of this idea proved to be successful and led to a publication in 1973 [16]. There, as the result of collaboration with colleagues, assistants and students, limiting absolute temperature ratios were given for the onset of instability, as a function of the (absolute value of the) Stokes parameter. Unfortunately, the complications caused by the use of an expansion procedure for E (and f*) that did *not* take advantage of an "analytic approximation" makes the understanding of the results of this study rather difficult. We have now a plan that promises simplifications but we will not pursue it here to its full completion.

The use of the new plan is contingent upon finding first the position of the temperature jump which is the most "dangerous", meaning that it leads to the smallest ratio of absolute temperatures needed for instability. The results of our 1973 paper indicated that for a tube with one open end, this jump position is very nearly at the midpoint, $\ell = L/2$. Now we will have a proof for this result.

For the case with two closed ends, Müller [12] [15] found that the most dangerous position of the temperature jump is located at the point where the ratio of hot to cold tube lengths is 2:5. This is not too far from a more simple-minded 1:3, which one could guess from the 1:1 for the open tube, assuming that the added cold length would have negligible effect on the most dangerous position. This guess would bring us quite close to the true extremum.

13. The "actuator disc" model

We now consider a new approach that begins with the formulation of a result of the analysis presented thus far which we think is the most important physically. We will show that while this proposition is not "exact" in the framework of the present study, it will remain "accurate" in the sense that it changes assumptions made thus far only within limits compatible with our "model". Its main advantage will be more transparency and thus a better understanding for further useful approximations.

The central proposition for the new start is that at the jump position of the temperature, the flux remains the same, $q_1 = q_2$. By use of the fundamental equation (69) this means

$$(1 - f_1)E_1 a_1^2 \left.\frac{dp}{dx}\right|_1 = (1 - f_2)E_2 a_2^2 \left.\frac{dp}{dx}\right|_2 \qquad (77)$$

This can be interpreted by stating that at the position $x = \ell$, we "model" the flow by an "actuator disc". In the inviscid limit when the f 's vanish and the E's are equal, the relation states that the quantity $(1/\rho_m)(dp/dx)$ is the same on both sides, i.e., the velocities are continuous. The ratio E_2/E_1 implies the calculation of the integral in the expression (65) for E between the limits given by the temperatures on the two sides of the jump. Thus these factors are the physical expression of our contention that we model an actuator disc. Finally, the friction factors $1 - f$ appear in (77). This can be interpreted as the influence of the velocity profile shapes on the flux.

The effect of the E-function is expressed, in good approximation, as a power m of the ubiquitous factor 1 – f. We have to accept this too as a quirk of nature; it is related to the one noted previously.

There is actually no compelling reason to expect that the actuator disk model will lead to exactly the same frequency relation as the one postulated in the previous section, in the equations (71) to (75). We had a rationale for the establishment of those relations. To compare their statement with the new condition (77), we first have to complete (77) by statements applicable for the rest of the system. It will be shown that the statement of (77) does not exactly agree with the previous postulates, but the difference will lie well within the limits of the uncertainties of our modeling. We will therefore refrain from attempts to explain those small differences and take full advantage of the simplifications offered by the new approach. We believe that our guess that led us to (77) was essentially correct.

To proceed we calculate the pressure gradient on the hot closed side of the disk from the exact isothermal equation (48) for p. We make use of the accepted simplification $(\gamma - 1)f^* = cf$:

$$\left.\frac{dp}{dx}\right|_2 = \frac{(1 + c f_2)\omega^2}{(1 - f_2)a_2^2} \int_{L-\ell}^{L} p\,dx \tag{78}$$

With the pressure $p = p_2$ from (71) one finds

$$\int_{L-\ell}^{L} p\,dx = p(\ell)\frac{\sin k_2(L - \ell)}{k_2 \cos k_2(L - \ell)} = p(\ell)(L - \ell)\frac{\tan \lambda_2}{\lambda_2} \tag{79}$$

This equation defines a complex wave number

$$\lambda_2 = k_2(L - \ell) = \frac{1 + cf_2}{1 - f_2}\frac{\omega(L - \ell)}{a_2} \tag{80}$$

in the hot part. The results thus far introduced in (77) give

$$(1 - f_1)E_1 a_1^2 \left.\frac{dp}{dx}\right|_1 = (1 + c f_2)E_2 \omega^2 p(\ell)(L - \ell)\frac{\tan\lambda_2}{\lambda_2} \tag{81}$$

In the last step, we restrict our attention to the tube with an open end at x = 0, which is the configuration most susceptible to oscillation. Consistent with the pressure distribution postulated in (71) is the relation

$$\left.\frac{dp}{dx}\right|_1 = \frac{p(\ell)}{\ell}\lambda_1 \cot\lambda_1 \tag{82}$$

where [see (72)]

$$\lambda_1 = k_1\ell = \frac{\omega\ell}{a_1}\left[\frac{1 + c f_1}{1 - f_1}\right]^{1/2} \tag{83}$$

Inserting (82) in (81) makes the whole expression divisible by $p(\ell)$, confirming that we have a homogeneous solution. The final result is rewritten in the form

$$\frac{\omega^2 \ell(L - \ell)}{a_1^2} = \frac{(1 - f_1)E_1}{(1 + cf_2)E_2}\frac{\lambda_1}{\tan\lambda_1}\frac{\lambda_2}{\tan\lambda_2} \tag{84}$$

From equations (78)-(79) and (82), we find that we can give a formula for the ratio of the pressure gradients on the two sides of the jump:

$$\left.\frac{dp}{dx}\right|_2 = \frac{1+cf_2}{1-f_2}\,\frac{\omega^2\ell(L-\ell)}{a_2^2}\left.\frac{dp}{dx}\right|_1 \qquad (85)$$

Elimination of $\omega^2\ell(L-\ell)$ between (84) and (85) and re-use of the relations (72) leads to an equation almost identical to the original frequency relation (73). The only difference is that to obtain the results that were already established in 1969, we should have found $\lambda_2 = \omega(L-\ell)/a_2$ instead of (80). However, it was noted ever since this type of theories were established that the factor $\tan\lambda_2 / \lambda_2$ can always safely be replaced by 1, causing only negligible errors.

It is now suggested that similarly, to set $\lambda_1 \cot\lambda_1 = 1$ in (82) is also an excellent approximation for a wide range of applications. The final formula is then, including the accepted simplifications on the E-function:

$$\frac{\omega^2\,\ell(L-\ell)}{a_1^2} = \frac{(1-f_1)E_1}{(1+cf_2)E_2} = \frac{(1-f_1)^{1+m}}{(1+cf_2)(1-f_2)^m} \qquad (86)$$

To reap the full benefits obtainable from these results, we have to abandon a simplifying assumption which was an important part of the analysis thus far, namely, that all generators of the wall surface are parallel to the x-direction. This is to be replaced to by the "Webster rule" [17] that requires that the product of the flux q times the cross-sectional area remains constant along the tube, a rule already anticipated and used by Rayleigh in many important applications. Now if we replace in (86) the factor $L-\ell$

by the volume V of the hot part divided by the area A_1 of the cold part, then we implement the Webster rule as applied by Rayleigh, and indeed, $(L - \ell) = V/A_1$ changes (84) *in the inviscid limit* to Rayleigh's classical formula for the frequency of oscillations in a Sondhauss tube [10, §303, §309, §321i]. The close connection of the mechanisms of oscillations for the Sondhauss and the Taconis tubes is herewith established. More details on the Sondhauss tube solution are given in [18].

It can be verified that the strongly simplified formula (86) still can describe some important features of the constant cross-section "Taconis" tube. We use the abbreviation

$$\frac{\omega^2\,\ell(L-\ell)}{a_1^2} = \lambda_c^2 \tag{87}$$

The expansion of (86) gives the following relation between the values of f on the two sides of the disc:

$$\lambda_c^2\left[1+(c-m)f_2+\frac{1}{2}m(m-1)f_2^2-+\ldots\right]= \\ 1-(1+m)f_1+\frac{1}{2}(1+m)mf_1^2-+\ldots \tag{88}$$

It is seen that when Kramers' constant $c - m$ vanishes, then in leading order, it is the quadratic term in f_2 on the hot side that has to be balanced by the linear term f_1 on the cold side in such a way that the relation (88) is real (positive or negative) upon multiplication by any real factor. Continuation of this procedure verifies the trends (i.e., power laws) of pertinent parts of the stability curve but does not give reliable numbers.

14. Conclusions

In retrospect, we can try to fit the results of this study in our overall view of the physics of simple gases. Then we can state that Kirchhoff's original plan to find linear homogeneous equations to describe the classical behavior of wave propagation has essentially succeeded in including the modest "thermodynamics" of the simplest possible case, the monatomic gas. In this very weakly damped realization of a classical "ensemble", Kirchhoff's original theory gives accurate fourth-order spatial relations which have, however, limited applicability. In the present study we have used extended linearization by introduction of one additional assumption which can be rationally supported for phenomena where only thin viscous boundary layers are present. Then second-order relations are found which are highly accurate in all isothermal situations. In non-isothermal cases simple problems can be solved but only for carefully constructed special boundary- and initial conditions

Unchanged from the original rules set forth by Kirchhoff was our insistence to restrict our attention to linear-homogeneous solutions. In the revision of the fundamentals of this theory presented here it was found that in non-isothermal flow situations, the use of a discontinuous temperature model is a prerequisite for the acceptability of the results for some special selected physical applications. Although we started with the general energy equation of the ideal gas, we did not need to invoke the energy equation *again* to clarify the energetics of the resultant motion.

If continuous stratification is found in nature or assumed in theory, discontinuous models can actually be very useful. Continuous stratified equilibrium situations need, in the absence of gravity (and of magneto-hydrodynamic effects), the introduction of heat sources and sinks for their maintenance. It is known that no thermal balance is complete without having the source-sink distribution *inside* the limits of the thermodynamics system considered. In the absence of such an analysis, the E-function can be used successfully as ersatz energy equation in an artificial world where undefined effects maintain a permanent temperature jump. Actually in some close realization of a sharp temperature change on a geophysical scale, the E-function could be a useful tool in the consideration of questions how mechanical work can be extracted from such a given situation. Necessary condition for applicability is strict linearity. Then the theory can be further augmented by consequences obtained by analytic continuation of the results discussed here. In this respect, the presentations of this paper are not complete. The use of analytic continuation was proposed in connection with the preparation of the 1973 paper [17] by Peter Monkewitz, then assistant at the ETHZ and now professor at the EPFL (Lausanne). The idea was extended to possible extremes in the dissertation of Müller [12], and it is probable that arguments used there involve extension of boundary layer approximations beyond their range of validity.

Actually linear-homogeneous solutions with negative damping are ephemeral by nature, and some new mechanism has to be recognized if it is intended to explain limiting amplitudes. Experience has shown that modes can undergo radical changes even at small amplitudes. An interesting

exception in the outward appearance of thermoacoustic phenomena is found when investigating oscillations in a U-tube filled with helium gas, closed at both ends. The tube ends are at room temperature while its midpart is submerged, e.g., in a liquid helium bath. The basic mode is antisymmetric with $p' = 0$ exactly at the midpoint and *not* influenced by the non-linear van Wijngaarden-effect at an open end. This configuration was used in the experiments of Yazaki, Tominaga and Narahara [19] who noted the extremely high pressure amplitudes in the resultant steady state, of the order of almost an atmosphere. The purpose of the experiments, however, was the observation of the onset of oscillations, and with the full realization of the ideal boundary- and end conditions, very satisfactory agreements with the linear theory of the Taconis tube were observed.

In this case, the appearance of the mode does not change much with the amplitude and it was conjectured that the known added secondary circumferential motion due to the bend in the U-tube contributes substantially to the damping. This was shown to be true by experiments in Zurich. Adding an extra full turn to the tube length substantially reduced the resultant final amplitude. Experiments by Olson and Swift [20] on this subject in Los Alamos were complemented by interesting theoretical considerations on the damping of oscillating flows in straight and bent tubes.

On the other hand, in an experiment with the non-linear van Wijngaarden end-effect present in a tube with an open end, the *onset* of oscillations should follow, nevertheless, the linear theory. In this case the end-effect will become dominant only after the amplitude has reached a

sufficiently high value so that it determines the limit cycle. Experience shows that this happens without noticeable influence on the frequency. A series of experiments by Rott and Zouzoulas [21] in a Sondhauss-type tube with nitrogen-gas filling confirmed this view of the end-effect and indicated that the non-linear effects of sudden cross-sectional changes fall into the same category. This shows that the Webster rule, which was already discussed, is the proper initial assumption for onset problems.

We think that to ask how steep an imposed discontinuous temperature distribution has to be so that the model can be used is asking a wrong question. In practice, temperature variations distributed over very many tube diameters can be treated with the discontinuous approximation, which is proven to provide a worst-case scenario. However, for modeling problems where the Webster rule is used, the actuator disk limit can be applied only at tube sections without area change.

In spite of some limited successes, however, there is no reason to assume that linear theory and the attendant discontinuous model is the only or the best possible means of solving thermoacoustic problems. In particular, one must look for non-linear mechanisms which can override the results of linear effects.

. 15. "Next"

To continue the analysis by dealing with non-linear effects, one is required to disregard the restriction of the subject matter expressed in the title of this paper. To end the analysis at this point gives, however, a lopsided view of the problem and of the possibility of finding further classes of solutions. As a compromise, we propose to give here a concise and mostly qualitative summary of what should be the subject of a next paper, which would deal with non-linear effects.

Generally, the application of a small external amplitude to an oscillatory system is an indispensable tool for its exploration. It yields the frequency response of the system, one of its most fundamental characteristics. This is also the problem already noted here as the one originally treated by Chester [8] who concentrated his efforts to the description of the part of the resonant mode which is dominated, for sufficiently high Reynolds numbers, by the appearance of shocks. However, the *first harmonic* content of these modes connects smoothly to the shock-free solutions, when the zone with shocks is approached with frequencies below resonance. This can be seen in more detail in the work of Keller [22] [23] on this subject. Thus to find the frequency response of this mode can give important insights in the structure of flow oscillations. Such a probe leads to steady state responses which are proportional to the square of the input amplitude, for small amplitudes.

The problem thus stated was treated in Zurich in a thesis prepared

under the direction of Professor Hans Thomann by Merkli [24]. Thomann proposed to measure the distribution of the heat transfer to the walls of a resonance tube. A straight tube with a carefully designed sinusoidal piston drive at one end was placed inside a second tube made of a material with low heat conductivity, coaxial to the actual resonance tube. The narrow air-gap between the tubes was first brought to full isothermal equilibrium before the oscillations in the resonance tube were started, and after start-up the transient temperature changes were recorded by thermocouples in the wall of the outer tube.

Considering now the steady state, it is found that the experiment of Merkli and Thomann realizes for a fully heat-insulated tube a situation where a quantity is conserved along the tube, for which the expression "enthalpy flux" became generally accepted. This statement is exact but it is thought that it is highly accurate even if it is restricted to the consideration of the gas inside the tube, assuming that the heat content of the massive wall is only negligibly affected in the direction parallel to the tube axis.. While this assumption turned out to be untenable in the long run, it appeared harmless when first made and will be used here. Then the total flux H of the enthalpy c_pT in a given tube cross-section is given in leading approximation by the following exact expression for its time-average,

$$\overline{H} = \int \rho_m c_p \overline{T'u}\, dA \qquad (89)$$

involving u and the first-order temperature disturbance T'. This is a direct consequence of the First Law [24] [25]. Merkli & Thomann discussed the statement of this equation when adjusted to the inhomogeneous isothermal

half-wavelength acoustic solution adjusted to the end conditions of a resonance tube. The heat transferred at the walls shows an extraordinary steep dependence on the Prandtl number, in good agreement with both theory and experiment. The axial distribution of the enthalpy flux can be further discussed with the help of the second-order energy equation (see also [9]). The axial flux, which is theoretically symmetrical with respect to the center of the tube, is directed in the wall layer *towards the nodes*, for values of the Prandtl number below about 3, i.e., practically always. The return circuit is in the core. The flux in the wall layer cools the antinode; $\sigma = 1$ is a critical value. For $\sigma < 1$, i.e., for all simple gases, there is a cooling effect relative to the initial equilibrium temperature. However, for vanishing σ, the same kind of divergence of the results is found that was already anticipated in previous sections.

It is noted that in the solution of Merkli & Thomann, the average temperature remains unchanged and that the second order motion creates a separation effect between hot and cold. There must be a close relation of this effect to Rayleigh acoustic streaming which is not yet explored.

The results of the Merkli-Thomann study opened the possibility of a total rethinking of the thermoacoustics problem. In competition to the "ETH Zurich" group, a new way of thinking about these problems was started by a group in Los Alamos NM. We try to pinpoint the essential new ideas next.

16. Other developments

The new ideas to be discussed now seem to fit perfectly into the formalism of the analysis as presented thus far yet represent a radical change in the underlying philosophy in response to the basic question: what is the problem and what is the solution ? Important publications that illustrate this point are (i) a paper published in 1983 by Wheatley, Hofler, Swift & Migliori [26], (ii) the (unpublished) 1986 dissertation of Hofler [27] , and finally (iii) the 1988 survey paper on "Thermoacoustic engines" by Swift [28]. This last paper contains in its "Appendix" a program for the solution of thermoacoustic problems which does not emphasize sufficiently its novel aspects. This is to be discussed here.

Swift posits at the beginning of his analysis the same linear equation (59) for p' that was derived here but proposes to consider it as the expression of two coupled complex equations for p' and dp'/dx [his equation (A31)]. This corresponds to the change of the second-order equation (59) to two first-order equations but Swift leaves the temperature distribution function T_m in (59) unchanged, and declares it as the fifth (real) unknown together with the two complex (= four real) unknowns p' and dp'/dx. The solution of this system is to be compared with the expression that it yields for the enthalpy flux. The thermoacoustic problem is solved when the assumed T_m leads to the desired enthalpy flux, upon its evaluation with the help of the relation (89). This flux must remain a constant after the point of the heat input (for which no theory is given) in a heat-insulated tube.

It is hard to find fault with this scheme except that it implies that steady state solutions exist in which the sound speed varies continuously. The previously presented efforts in this paper were justified by the idea that in thermoacoustics, piecewise constant regions of the sound speed must be found. This led to the discontinuous model and the E-function. These features are absent in the Swift model. However, we question whether it is permissible to postulate a continuously varying sound speed without including in the analysis the thermodynamics of the container which maintains such a steady state. We might even question this assumption if we do not know how this state can be reached.

Actually, there is a very fundamental difficulty which occurs when the scheme proposed by Swift is applied. There is a difference in the development of the behavior of thermoacoustic devices depending on the way the energy input occurs *first*. If a thermal energy distribution is established first while the gas motion is in the start-up phase, then I believe linear theory is the right way to attack the problem. If, however, we deal with a device where the initial enthalpy input is in form of mechanical energy, then the scheme fails as non-linear theory is needed at the very beginning of the process. Experience with so-called "thermoacoustic prime movers" shows that while initially the motions appear to follow predictions by linear theory, the continuation definitely requires the understanding of non-linear effects . What is wrong, then, is to rely on the linear equation (59) to describe any steady state motion that emerges. To put it bluntly, equation (59) is in general wrong. It is applicable only in particular situations.

Actually it is not hard to realize that a temperature gradient in a long gas-filled tube can be maintained only with the help of solid conduction in the walls. If the gas flow in the tube is oscillating, acoustic effects must be secondary to heat exchange effects with the wall, because there are no steady state acoustic solutions with a variable sound speed.

In 1995, a very simple solution for a system formed by a conducting tube with oscillating gas filling was presented by Bauwens [29]. It seemed that a connection with thermoacoustics should remain simple, but this hope was not fulfilled. In the solutions given *before* Bauwens, the periodic heat exchange with the wall was obviously not connected correctly with the longitudinal conduction. This part could probably be improved by better treatment in thermoacoustics. However, I believe that there is an element in Bauwens' approach that precludes a simple connection to thermoacoustics "the way we know it". Bauwens works with the mass flux instead if the volume flux. The initial averaging that permitted the establishment of a relation between pressure gradient and volume flux is not valid any more in the Bauwens theory and is replaced by a more complicated law. Discussion of further details would necessitate a presentation of Bauwens' ideas and we do not feel up to this task. Bauwens himself noted immediately the connection of his ideas with a classical theory of the regenerator, given by Schmidt [30] in 1871. In a later publication [31] he established the connection to a gas flow model which needed expansion in two small parameters, essentially heat conductivity and pressure gradient. The complications are substantial and increase further when Bauwens attempts, by use of multiple time scales, to establish how the equilibrium in a basic

simple regenerator is established in time. His aim is to show that the same final stage is reached for arbitrary initial conditions [32]. The paper is hard to follow but there seems to be no way back to "simple" thermoacoustics.

Finally, a contribution to Swift-style thermoaoustics is noted which was presented by Arnott, Bass & Raspet [33] in 1991. They purportedly intended to explain acoustics in "pores", a notion that was not completely defined. (This was also apparently an excuse to introduce a complete set of new notations.) Actually the use of the similarity parameter in fluids, i.e., the Reynolds number or a variant adjusted to unsteady flow like the Stokes parameter, permits the treatment of acoustics problems on any scale. Thus a theory suitable for pores should also be applicable to sound propagation in culverts. Interesting observations on this subject, in a paper entitled "Culvert whistlers", was published by Crawford [34] in 1971, which led to the consideration of effects that were not even mentioned in this paper: the question of transverse oscillations. It is recalled that these were included in the theory originally presented by Kirchhoff. Our view is that to consider all transverse oscillations as indirect boundary layer "Taconis" effects *only* is untenable if we admit mechanical introduction of enthalpy into the system.

It is of interest that in a more recent paper by Raspet, Brewster & Bass [35] ideas connected with the E-function reappeared. I think that this approach has been discussed here in such detail that its possibilities and its shortcomings can be well understood.

Abstract. The effect of heat conduction on sound propagation in simple gases, first treated by Kirchhoff in 1868, is reconsidered and variations to the linear-homogeneous Kirchhoff theory on sound propagation in ducts are established. The new inquiry in these old problems is motivated by the extension of the Kirchhoff theory to sound propagation in ducts wherein a temperature gradient along the axis is maintained. The first such problem was solved by Kramers in 1949. He succeeded in constructing special linear solutions in ducts where two parts of a gas column are considered, each at constant temperature, and with the ratio of the absolute temperatures in the two regions so high that the time needed to traverse the hot part becomes negligible compared to the time of the wave spent in the cold part.

This admittedly artificial scheme was simplified in a paper I published in 1969 (Z. angew. Math. Phys. **20**, 230) and is revised and clarified here again. This involves a revision of the Kirchhoff theory as it uses the pressure instead of the density as the primary state variable to establish the connection between the energy- and the momentum equations. The clarification of this point is the main motivation for the present paper. Unchanged remain the limits set to the theory by linearization, and the use of an attendant discontinuous model of the temperature distribution.

Final remarks are concerned with the inhomogeneous non-linear frequency response investigated by Merkli & Thomann (1976), the solution scheme proposed by Swift (1988), and the basic ideas of a non-linear theory by Bauwens (1995), which show the way for possible progress.

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